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Classical and quantum consistency of the DGP model

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Abstract: We study the Dvali-Gabadadze-Porrati model by the method of the boundary effective action. The truncation of this action to the bending mode π consistently describes physics in a wide range of regimes both at the classical and at the quantum level. The Vainshtein effect, which restores agreement with precise tests of general relativity, follows straightforwardly. We give a simple and general proof of stability, i.e. absence of ghosts in the fluctuations, valid for most of the relevant cases, like for instance the spherical source in asymptotically flat space. However we confirm that around certain interesting self-accelerating cosmological solutions there is a ghost. We consider the issue of quantum corrections. Around flat space π becomes strongly coupled below a macroscopic length of 1000 km, thus impairing the predictivity of the model. Indeed the tower of higher dimensional operators which is expected by a generic UV completion of the model limits predictivity at even larger length scales. We outline a non-generic but consistent choice of counterterms for which this disaster does not happen and for which the model remains calculable and successful in all the astrophysical situations of interest. By this choice, the extrinsic curvature $K_{\mu\nu}$ acts roughly like a dilaton field controlling the strength of the interaction and the cut-off scale at each space-time point. At the surface of Earth the cutoff is ~ 1 cm but it is unlikely that the associated quantum effects be observable in table top experiments.

Keywords: Extra Large Dimensions, Classical Theories of Gravity.

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Contents

1. Introduction

In recent years several attempts have been made to formulate a theoretically consistent extension of General Relativity that modifies gravity at cosmological distances while remaining compatible with observations at shorter distances. Well known examples in the literature are models of massive gravity [1, 2], the GRS model [3], the DGP model [4], and the newborn model of 'ghost condensation' [5]. These attempts are motivated both by the appealing theoretical challenge they represent, and, perhaps more interestingly, by the experimental indication that something strange is indeed happening at very large scales: the expansion of the Universe is accelerating. This fact can be explained, as it is usually done, by invoking a small cosmological constant, or some sort of 'dark energy'. Nevertheless, it is also worth considering the possibility that at cosmological distances gravity itself is different from what we experience at much smaller scales, in the solar system. This perspective could perhaps offer a new direction for addressing the cosmological constant problem (see for instance ref. [6]).

In the present paper we concentrate on the DGP model. In this model there exists a critical length scale $L_{\rm DGP}$ below which gravity looks four dimensional, while for larger distances it weakens and becomes five dimensional. To avoid conflict with observations, the parameters of the model can be adjusted to make $L_{\rm DGP}$ of the order of the present Hubble horizon H_0^{-1}. One cosmological solution of DGP is an ordinary Friedmann-Robertson-Walker Universe that, even in the absence of a cosmological constant, gets accelerated

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at late times by the weakening of gravity [7, 8]. Moreover, even though the length scale of gravity modification is cosmological, one still expects small but non-negligible effects on solar system astrophysical measurements [9, 10]. This makes the model extremely interesting from the phenomenological point of view.

In spite of these appealing features, the model is plagued by some consistency problems, as pointed out in ref. [11]. There it was shown that the theory has strong interactions at a tiny energy scale

$$\Lambda \sim \left(\frac{M_P}{L_{\rm DGP}^2}\right)^{1/3}, \tag{1.1}$$

which corresponds to wavelengths of about 1000 km (a qualitatively similar, but quantitatively different, result was found in ref. [12]). This means that below 1000 km the model loses predictivity, and one cannot perform sensible computations without knowing the UV completion of the theory. In the presence of curvature the situation seems even worse. Indeed in a non-renormalizable theory like the one at hand, quantum corrections necessarily imply the presence of an infinite tower of higher dimensional operators suppressed by inverse powers of the interaction scale Λ. A 'generic' choice of this tower of operators disrupts the classical computations of the original DGP model [11], making it practically uncalculable, whenever the Riemann curvature is bigger than $\mathcal{O}(1/L_{\rm DGP}^2)$. In practice the unknown UV completion of the model dominates physics above the tiny curvature $1/L_{\rm DGP}^2$. In the case of a spherical source in asymptotically flat space this happens below a distance from the source given by the so called Vainshtein length [13, 14]

$$R_V \sim (R_S\, L_{\rm DGP}^2)^{1/3}, \tag{1.2}$$

where R_S is the Schwarzschild radius of the source. For instance, for the Sun R_V is about 10^{20} cm! Apparently one cannot compute the gravitational potential of the Sun at distances shorter than 10^{20} cm. A completely equivalent problem has been pointed out in the case of massive gravity [15]. It should be stressed that, in the purely classical DGP model, R_V is precisely the length below which classical non-linearities in the field equations eliminate the phenomenologically unwanted effects of the extra polarizations of the graviton [13, 14]. The latter are usually referred to as the van Dam-Veltman-Zakharov discontinuity [16]. These non-linearities are therefore a virtue of the classical theory, but upon quantization they suggest the presence of many more uncontrollable interactions.[1]

Finally, the model possesses negative energy classical solutions, which signal the presence of instabilities already at the classical level. These negative energy solutions have however a typical curvature larger than $1/L_{\rm DGP}^2$, so that their true presence, or absence, is fully dependent on the UV completion.

[1]In a recent paper ref. [17] a regularized version of DGP was analized, in which instead of having a localized 4D Einstein-Hilbert term, the brane is surrounded by a thin 'dielectric' with reduced Newton's constant. It was found that such a model shares with the original DGP the crossover between the 4D and the 5D behaviour of the graviton propagator, but it lacks all the (good or bad) effects due to large non-linearities in the scalar sector: there is no strong interaction scale Λ below M_5, no Vainshtein effect, and therefore the model *is* plagued by the van Dam-Veltman-Zakharov discontinuity.

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In this paper we try to understand to what extent all these results really threaten the consistency and the predictivity of the model. In spite of the above difficulties we will be able to draw fairly optimistic conclusions. In practice we will show that one can consistently assume a UV completion ($\equiv$ tower of higher dimensional operators) where the theory can be extrapolated down to distances significantly shorter than $1/\Lambda$. Though our choice is consistent, in the sense that it is stable under RG evolution, it may at first sight look unnatural from the effective field theory view point. We will try to argue that perhaps it isn't.

In the DGP model our world is the 4D boundary of an infinite 5D spacetime. One can integrate out the bulk degrees of freedom and find an "effective" action for the 4D fields. This was explicitly done in ref. [11], where it was found that from the 4D point of view, beside the ordinary graviton, there is an extra scalar degree of freedom π that plays a crucial role. This is essentially a brane bending mode contributing to the extrinsic curvature of the boundary like $K_{\mu\nu} \propto \partial_\mu \partial_\nu \pi$. It is this scalar that interacts strongly at momenta of order Λ. Indeed there exists a limit in which the strong interaction scale is kept fixed but all other degrees of freedom (namely the graviton and a vector N_μ) decouple. This means that in order to grasp the interesting physics related to the strong interaction we can restrict to the π sector, which represents a consistent truncation of the theory. We will see that all the interesting features (good or bad) of the DGP model can be traced to the dynamics of this scalar π. For our purposes we can therefore forget about the 5D geometric setup of the model and study a scalar theory with a specific cubic interaction, namely $\frac{1}{\Lambda^3}(\partial\pi)^2 \Box\pi$. This largely simplifies the analysis.

We start in section 3 by studying classical field configurations in this theory. For any given distribution of matter sources, the field equation for π turns out to be a quadratic *algebraic* equation for the tensor of its second derivatives $\partial_\mu \partial_\nu \pi$. Like ordinary quadratic equations, it possesses doublets of solutions: for any solution $\pi^+(x)$, there exists a corresponding 'conjugate' solution $\pi^-(x) \neq \pi^+(x)$. By studying the stability of a generic solution against small fluctuations, one discovers that this conjugation exactly reverses the sign of the kinetic action of the fluctuations. This means that if $\pi^+(x)$ is a stable solution, its conjugate $\pi^-(x)$ must be unstable. It is therefore not surprising that classical instabilities were found in ref. [11].

Does the existence of these unstable solutions make the model inconsistent at the classical level? We argue that this is not the case: the classical stability of the π system is guaranteed for a well defined class of energy-momentum sources and boundary conditions at spatial infinity, as we prove explicitly in section 4. In particular, we show that for any configuration of sources with positive energy density decaying at spatial infinity, and small pressure (smaller than $\rho/3$), the solution which is trivial at spatial infinity is stable everywhere.[2] Of course the conjugate configuration, which instead will diverge at infinity, will be unstable. The point is that the sets of stable and unstable solutions are disconnected in field space, they cannot 'communicate'. It suffices that a solution is locally stable at one space-time point to ensure its global stability in the whole Universe, because there

[2]If the sources have relativistic velocities an additional hypothesis is required, see section 4.

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exists no continuous path of solutions that connects the stable and the unstable regions in field space. Unstable solutions simply correspond to the 'wrong' choice of boundary conditions.

Unfortunately nothing prevents some interesting solutions from having the wrong boundary behaviour: it is the case of the de Sitter solution, which describes the self-accelerated Universe we mentioned above. In ref. [11] it was pointed out that such solution is plagued by ghost-like instabilities. In section 5 we check this result in our formalism, indeed finding that the self-interacting dynamics of π is responsible for the existence of the de Sitter solution. The latter turns out to be nothing but the conjugate of the trivial configuration $\pi(x) = 0$, and as such must be unstable.

In section 6 we study the solution for π in presence of a spherical heavy source. We reproduce in the π language the results of refs. [18, 19, 9, 20], namely that the correction induced by the DGP model scales like $(r/R_V)^{3/2}$ relative to Newton's law. As stressed in ref. [9], the sign of this correction depends on the cosmological phase, i.e. on the behaviour of the Universe at infinity. In accordance with the general proof given in section 4, we explicitly check that fluctuations around this solution are stable. Also in this case the interesting features of DGP are encoded in the non trivial dynamics of the field π, in a simple and transparent way.

The results of sections 4–6 imply that for any astrophysical source and for a wide and relevant class of cosmological solutions no classical instability develops in the π field. Finally, in section 7, we discuss the problem of quantum corrections. We have already mentioned that a generic choice of higher dimension operators leads to uncalculability when the Riemann curvature is bigger than $\mathcal{O}(1/L_{\rm DGP}^2) \sim H_0^2$. We look at this problem in more detail. We characterize higher order corrections, distinguishing between a classical and a quantum expansion parameter. We notice that the large effects of the counterterms which were pointed out in ref. [11] are more related to the classical expansion than to the quantum one. We clarify this, by discussing first the structure of the 1-loop effective action. We show that the terms which are truly needed for unitarity and consistency, and which are associated to logarithmic divergences, do not grow in the regions of large curvature. In a sense this follows simply from the fact that the classical theory is well defined, i.e. there are no ghosts. Inspired by this simple example we then proceed to give a general and consistent characterization of the class of counterterm lagrangians which do not destabilize the naive classical results. Our claim is based on a remarkable property of the tree level action for π. The large curvature backgrounds around which one seemingly looses calculability correspond to a classical background $|\partial_\mu\partial_\nu\pi_{cl}/\Lambda^3| \gg 1$. Around such a background the quantum fluctuation $\varphi = \pi - \pi_{cl}$ receives a large positive contribution to its kinetic term

$$\mathcal{L}_{\rm kin} = Z_{\mu\nu}(\partial^\mu\varphi)(\partial^\nu\varphi)\,, \tag{1.3}$$

where $Z_{\mu\nu}$ is a non singular matrix proportional to $\partial_\mu\partial_\nu\pi_{cl}/\Lambda^3$ itself. The interaction lagrangian remains however the same: no large factors there. The large kinetic term then suppresses interactions (as seen for instance by going to canonical fields). At each spacetime point x one can define (suppressing spacetime indices) a naive local scale $\tilde{\Lambda}(x) =$

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$\Lambda\sqrt{Z(x)} \gg \Lambda$, which is a function of the background field π_{cl} itself. Our basic result is then the following. If the counterterm action depends on π and Λ *only* via the running scale $\tilde\Lambda$ and its derivatives, then the classical DGP action can be used down to the distance $1/\tilde\Lambda(x)$ at each point x. This implies, in particular, that the Vainshtein solutions within the solar system are always fine. On the other hand at the Earth surface one finds $\tilde\Lambda \sim 1\,\mathrm{cm}^{-1}$. In principle this could be worrisome, or interesting. In practice it is probably neither. This is because at the surface of Earth π is well decoupled from matter.

The counterterm structure we select must correspond to some specific UV completion for the physics at the 4D boundary. At the scale $\tilde\Lambda$ we expect new states to become relevant. This scale is determined by the background expectation value of the massless field π, or more precisely $\partial_\mu\partial_\nu\pi \propto K_{\mu\nu}$. Therefore the extrinsic curvature acts like a sort of dilaton controlling the scale of the new physics. A direct consequence of this property is that our ansatz can be motivated by an approximate scale invariance of the UV completion.

We conclude with an outlook on possible developments of our results.

2. Boundary effective action

The DGP model [4] describes gravity in a five-dimensional spacetime $\mathcal{M}$, with boundary $\partial\mathcal{M}$. The action is postulated to be purely Einstein-Hilbert in the bulk, plus a four dimensional Einstein-Hilbert term localized at the boundary, with different Planck scales,

$$S_{\rm DGP} = 2M_5^3\int_{\mathcal{M}} d^5x\sqrt{-G}\,R(G) + 2M_4^2\int_{\partial\mathcal{M}} d^4x\sqrt{-g}\,R(g) - 4M_5^3\int_{\partial\mathcal{M}} d^4x\sqrt{-g}\,K(g)\,, \quad (2.1)$$

where G_{MN} is the 5D metric, $g_{\mu\nu}$ is the 4D induced metric on the boundary, and we have added the Gibbons-Hawking term at the boundary in order to obtain the 5D Einstein equations upon variation of the bulk action [21]. Matter fields and a possible cosmological constant are supposed to be localized on the boundary. A special role is played by the length scale $L_{\rm DGP} = 1/m \equiv M_4^2/M_5^3$: below $L_{\rm DGP}$ gravity looks four-dimensional, while at larger length scales it enters in the five-dimensional regime. In order for the model to be viable, $L_{\rm DGP}$ must be huge, at least of the order of the present Hubble horizon.

In the following sections we will refer to the 4D boundary effective action obtained in ref. [11] by integrating out the bulk degrees of freedom. We summarize here only the main results. By using spacetime coordinates $\{x^\mu, y\}$, with the boundary sitting at $y = 0$, the bulk piece of the action can be rewritten in terms of ADM-like variables as

$$S_{\rm bulk} = 2M_5^3\int d^4x\int_0^\infty dy\sqrt{-g}\,N\left[R(g) - K^{\mu\nu}K_{\mu\nu} + K^2\right], \quad (2.2)$$

where $N = 1/\sqrt{G^{yy}}$ is the lapse, $N_\mu = G_{y\mu}$ is the shift, $g_{\mu\nu} = G_{\mu\nu}$ is the metric on surfaces of constant y, and the extrinsic curvature tensor is

$$K_{\mu\nu} = \frac{1}{2N}\left(\partial_y g_{\mu\nu} - \nabla_\mu N_\nu - \nabla_\nu N_\mu\right). \quad (2.3)$$

One then expands the 5D metric and all the other geometric quantities around a flat background, $G_{MN} = \eta_{MN} + h_{MN}$, and integrate out the bulk in order to obtain an effective

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action for the 4D fields living on the boundary. The final result, at the quadratic level, is

$$S_{\rm bdy} \simeq M_4^2 \int d^4x \left[\frac{1}{2} h'^{\mu\nu} \Box h'_{\mu\nu} - \frac{1}{4} h' \Box h' - m\, N'^{\mu} \Delta N'_{\mu} + 3m^2 \, \pi \, \Box \pi \right], \tag{2.4}$$

where Δ is a non-local differential operator, $\Delta = \sqrt{-\Box} = \sqrt{-\eta^{\mu\nu}\, \partial_\mu \partial_\nu}$, and the kinetic terms have been diagonalized by defining

$$h_{yy} = -2\Delta\pi \,, \qquad N'_\mu = N_\mu - \partial_\mu \pi \,, \qquad h'_{\mu\nu} = h_{\mu\nu} - m\, \pi\, \eta_{\mu\nu} \,. \tag{2.5}$$

By taking into account bulk interactions with higher powers of h_{MN}, one finds that the leading boundary interaction term is cubic in π, and involves four derivatives,

$$\Delta S^{(3)}_{\rm bdy} = -M_5^3 \int d^4x \, (\partial\pi)^2 \, \Box \pi \,. \tag{2.6}$$

The comparison of this interaction with the kinetic term of the π field in eq. (2.4) immediately shows that the π sector of the theory becomes strongly interacting at a very small energy scale $\Lambda = (m^2 M_4)^{1/3} = M_5^2/M_4$, which corresponds to a length scale of about 1000 km.

The boundary effective action we will use throughout the paper is the sum $S_{\rm bdy} + \Delta S^{(3)}_{\rm bdy}$. Notice that $\Delta S^{(3)}_{\rm bdy}$ is the leading boundary interaction term in a quantum sense: it gives the largest amplitude in low energy scattering processes, and the scale Λ associated to it is the lowest of all strong interaction scales associated to further interaction terms. These terms are schematically of the form [11]

$$\Delta\mathcal{L}_{\rm bdy} \sim M_5^3 \, \partial \, (N'_\mu)^p \, (\partial\pi)^q \, (h'_{\mu\nu})^s \sim m M_4^2 \, \partial \left(\frac{\hat{N}'_\mu}{m^{1/2} M_4} \right)^p \left(\frac{\partial \hat{\pi}}{m M_4} \right)^q (h'_{\mu\nu})^s \,, \tag{2.7}$$

where $\hat{N}'_\mu$ and $\hat{\pi}$ are canonically normalized, and $p + q + s \geq 3$. Eq. (2.6) corresponds to the term with $p = s = 0$ and $q = 3$.

However, in this paper we study classical field configurations, and from a classical point of view it is not immediately manifest why eq. (2.6) should be the most important interaction term, and why we should be allowed to neglect all the others. In fact, the strong interaction scale Λ has no meaning at the classical level (see discussion in section 7). Nevertheless, we will check explicitly in the two specific cases we will deal with (the de Sitter solution and the spherically symmetric solution) that indeed all other interaction terms are subdominant as long as the flat-space approximation is valid, i.e. as long as $|h'_{\mu\nu}| \ll 1$. This fact can be understood in a general although formal way. First, if $|h'_{\mu\nu}| \ll 1$, in eq. (2.7) we can stick to terms with $s = 0$. Then, suppose we have a compact source for our fields: the region in which space is nearly flat is the region well outside the Schwarzschild radius R_S. We can mimic to be in this region by formally sending R_S to zero, i.e. by decoupling four-dimensional gravity from the source. In this limit the solution for the metric is trivial, $h'_{\mu\nu} = 0$. However, we do not want to end up in a completely trivial configuration, so we want to preserve the self-coupling of the π field, as well as its interaction with the source: in terms of the canonically normalized $\hat{\pi}$ field, the cubic self coupling is unchanged if we

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keep $\Lambda = M_5^2/M_4$ fixed; in order to take into account the interaction with matter sources, we notice that the interaction lagrangian between the 4D metric perturbation $h_{\mu\nu}$ and the matter stress energy tensor $T_{\mu\nu}$ is, by definition, $\frac{1}{2}h_{\mu\nu}T^{\mu\nu}$. From the definition of π, eq. (2.5), we see that $\hat{\pi}$ interacts with matter via a term $\frac{1}{2M_4}\hat{\pi}T$. Therefore, if we take the formal limit

$$M_4, M_5, T_{\mu\nu} \to \infty\,, \qquad \frac{M_5^2}{M_4} = \text{const}\,, \quad \frac{T_{\mu\nu}}{M_4} = \text{const}\,, \tag{2.8}$$

we decouple 4D gravity while keeping the full lagrangian for $\hat{\pi}$ fixed. By 'full' we mean the sum of kinetic, cubic and source terms. We stress that when applying this limit to a spherical source of mass M, the Vainshtein radius $R_V^3 = R_S L_{\rm DGP}^2 = M M_5^6/M_4^3$ remains fixed, showing that we are focusing on the genuine non-linearities of the DGP model. The point now is that in the above limit all coefficients of further interactions, eq. (2.7), vanish,

$$m M_4^2\, \partial \left(\frac{\hat{N}'_\mu}{m^{1/2}M_4}\right)^p \left(\frac{\partial\hat{\pi}}{mM_4}\right)^q = \left(\frac{M_4}{M_5^2}\right)^q \frac{1}{M_5^{3p/2+q-3}}\, \partial\,(\hat{N}'_\mu)^p(\partial\hat{\pi})^q \to 0\,. \tag{2.9}$$

So far we neglected the dynamics of $\hat{N}'_\mu$, but the above result tells us that we were justified in doing so: $\hat{N}'_\mu$ does not interact directly with matter, and in the limit we are considering all its interactions vanish, thus decoupling it from all other degrees of freedom. In conclusion, we see that in the limit in which the metric perturbation can be neglected, $|h'_{\mu\nu}| \ll 1$, eq. (2.6) gives the dominant interaction term also for classical field configurations. Conversely our approximation breaks down when $|h'_{\mu\nu}| \sim 1$, like when approaching a black-hole horizon. In fact when this happens we also have $(\partial\hat{\pi}/mM_4) \equiv \partial\pi \sim 1$ (see section 6). Since π roughly measures the bending of the brane, this simply tells us that our approximation breaks down when the brane can no longer be treated as approximately flat in 5D. In the final section we will briefly explain what we expect to happen in this regime.

3. General properties of classical solutions

We now move to study classical solutions in presence of matter sources on the boundary. We restrict to the π sector, with cubic self-interactions and coupled to matter. As we showed in the previous section, this is a consistent truncation of the theory, in the sense that there exist a limit, eq. (2.8), in which all other degrees of freedom decouple, and all further interactions vanish.

From the results reported in the previous section we have that the full action for the π field in flat space is

$$S = \int d^4x \left[-3(\partial\hat{\pi})^2 - \frac{1}{\Lambda^3}(\partial\hat{\pi})^2\,\Box\hat{\pi} + \frac{1}{2M_4}\,\hat{\pi}T\right], \tag{3.1}$$

where $T = T^\mu{}_\mu$, and we canonically normalized π.

The first variation of S with respect to $\hat{\pi}$ gives the field equation

$$6\,\Box\hat{\pi} - \frac{1}{\Lambda^3}\Box(\partial\hat{\pi})^2 + \frac{2}{\Lambda^3}\partial_\mu(\partial^\mu\hat{\pi}\,\Box\hat{\pi}) + \frac{T}{2M_4} = 0\,, \tag{3.2}$$

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which can be rewritten as

$$3\,\Box\hat{\pi}-\frac{1}{\Lambda^3}(\partial_\mu\partial_\nu\hat{\pi})^2+\frac{1}{\Lambda^3}(\Box\hat{\pi})^2=-\frac{T}{4M_4}\,. \tag{3.3}$$

Notice that the terms involving third derivatives of $\hat{\pi}$ have canceled out, thus leaving us with an *algebraic* equation for its second derivatives. In terms of the tensor field $\tilde{K}_{\mu\nu}(x)=-\frac{1}{\Lambda^3}\partial_\mu\partial_\nu\hat{\pi}$ the above equation reads

$$\tilde{K}^2-(\tilde{K}_{\mu\nu})^2-3\tilde{K}=-\frac{T}{4\Lambda^3 M_4}\,, \tag{3.4}$$

where $\tilde{K}=\tilde{K}^\mu{}_\mu$. This result is easy to understand from a geometric point of view, by referring to the 5D setup of the model. The field equation that derives from varying the bulk action eq. (2.2) with respect to the lapse N is one of the Gauss-Codazzi equations,

$$R(g)+K^{\mu\nu}K_{\mu\nu}-K^2=0\,, \tag{3.5}$$

which is an algebraic relation between intrinsic and extrinsic curvature of the boundary. The intrinsic curvature $R(g)$ can be eliminated by means of the Einstein equations on the boundary,

$$4M_4^2\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\,R\right)-4M_5^3(K_{\mu\nu}-g_{\mu\nu}\,K)=T_{\mu\nu}\,, \tag{3.6}$$

thus obtaining an algebraic relation between the extrinsic curvature $K_{\mu\nu}$ and the matter stress-energy tensor. The extrinsic curvature is easily related to π by expanding eq. (2.3),

$$K_{\mu\nu}=m\left[-\frac{1}{\Lambda^3}\partial_\mu\partial_\nu\hat{\pi}\left[1+\mathcal{O}\left(\frac{1}{M_5}\cdot\frac{M_4}{M_5^2}\partial\hat{\pi}\right)\right]+\mathcal{O}\left(\frac{1}{M_5}\cdot\frac{M_4}{M_5^2}\partial\hat{h}'_{\mu\nu}\right)+\mathcal{O}\left(\frac{1}{M_5^{3/2}}\cdot\partial\hat{N}'_\mu\right)\right]. \tag{3.7}$$

In the limit we are considering π gives the dominant contribution, $K_{\mu\nu}\simeq-\frac{m}{\Lambda^3}\partial_\mu\partial_\nu\hat{\pi}=m\tilde{K}_{\mu\nu}$, and this leads precisely to our eq. (3.4). This remark also tells us that eq. (3.4) is more fundamental than its counterpart written in terms of $\hat{\pi}$, eq. (3.3). The latter is valid only in the limit of nearly flat space, while the former is a combination of a geometric identity and of the exact Einstein equations on the boundary, and as such is exact also in presence of large curvatures, as long as the classical theory makes sense. Of course in such a case the tensor $\tilde{K}_{\mu\nu}$ is the extrinsic curvature of the boundary in units of m, rather than the second derivative of $\hat{\pi}$.

Once a solution of eq. (3.4) for a given distribution of sources has been found, one may be interested in studying its stability. In order to do that, it is necessary to perturb $\hat{\pi}$ and expand the action up to quadratic order in the perturbation φ,

$$S_\varphi=\int d^4x\left[-3(\partial\varphi)^2-\frac{1}{\Lambda^3}\left(\Box\hat{\pi}\,(\partial\varphi)^2+2\,\partial_\mu\hat{\pi}\,\partial^\mu\varphi\,\Box\varphi\right)\right]. \tag{3.8}$$

By using the identity $\partial^\mu\varphi\Box\varphi=\partial_\nu[\partial^\nu\varphi\partial^\mu\varphi-\frac{1}{2}\eta^{\mu\nu}(\partial\varphi)^2]$, and integrating by parts, we find

$$S_\varphi=\int d^4x\left[-3(\partial\varphi)^2+\frac{2}{\Lambda^3}\left(\partial_\mu\partial_\nu\hat{\pi}-\eta_{\mu\nu}\Box\hat{\pi}\right)\,\partial^\mu\varphi\partial^\nu\varphi\right] \tag{3.9}$$

$$=\int d^4x\left[-3(\partial\varphi)^2-2\left(\tilde{K}_{\mu\nu}-\eta_{\mu\nu}\tilde{K}\right)\,\partial^\mu\varphi\partial^\nu\varphi\right]. \tag{3.10}$$

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This result deserves some comments. First, notice that the starting cubic interaction term involves four derivatives, but the resulting quadratic action for the fluctuation φ does not contain terms with more than two derivatives acting on the φ's. This remarkable feature makes the issue of stability more tractable, since it prevents the appearance of ghost states at high momenta, when higher derivatives contributions to the quadratic action could become important and destabilize the system. Second, the coefficients in S_φ depend only on the tensor $\tilde{K}_{\mu\nu}$, in an algebraic way. The fact that both the field equations and the condition for stability depend algebraically on $\tilde{K}_{\mu\nu}$ will play a crucial role in what follows.

A deeper insight into the structure of classical solutions comes from the following remark. For a given distribution of sources, eq. (3.4) is a second-order algebraic equation for $\tilde{K}_{\mu\nu}$. Instead of having a single solution, it is reasonable to expect doublets of solutions $\tilde{K}^{\pm}_{\mu\nu}$, like in ordinary quadratic equations. Moreover, one expects that in general the stability of the system depends on the solution chosen inside this doublet, since the quadratic lagrangian for the fluctuation φ, eq. (3.10), depends on $\tilde{K}_{\mu\nu}$. Even better, given the simple dependence of the overall coefficient of $\partial^\mu\varphi\partial^\nu\varphi$ in eq. (3.10) on $\tilde{K}_{\mu\nu}$,

$$Z_{\mu\nu} = -3\eta_{\mu\nu} - 2(\tilde{K}_{\mu\nu} - \eta_{\mu\nu}\tilde{K})\,, \tag{3.11}$$

one can straightforwardly invert this relation and express $\tilde{K}_{\mu\nu}$ in terms of $Z_{\mu\nu}$,

$$\tilde{K}_{\mu\nu} = \frac{1}{2}\eta_{\mu\nu} - \frac{1}{2}(Z_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}Z)\,, \tag{3.12}$$

where $Z = Z^\mu{}_\mu$. This is remarkable: now we can write our field equation, eq. (3.4), directly as an algebraic equation for the kinetic coefficient of the fluctuations,

$$\frac{1}{3}Z^2 - (Z_{\mu\nu})^2 = 6 - \frac{T}{2\Lambda^3 M_4}\,. \tag{3.13}$$

In the above equation the linear term in $Z_{\mu\nu}$ has canceled out, thus leading to a purely quadratic equation. This explicitly shows that if $Z^+_{\mu\nu}$ is a solution, so is its 'conjugate' $Z^-_{\mu\nu} \equiv -Z^+_{\mu\nu}$.

By definition eq. (3.10) now simply reads,

$$S_\varphi = \int d^4x\, Z_{\mu\nu}(x)\, \partial^\mu\varphi\partial^\nu\varphi\,, \tag{3.14}$$

which shows that a solution is locally stable against small fluctuations if and only if the matrix $Z_{\mu\nu}$, once diagonalized, has the usual 'healthy' signature $(+,-,-,-)$. (In the following section we will be more precise in defining the requirements that ensure the stability of the system.) Therefore, if a solution $Z^+_{\mu\nu}$ is locally stable, its conjugate $Z^-_{\mu\nu}$ is locally unstable. The kinetic terms of the small fluctuations around the two solutions are exactly opposite.

Coming back to $\tilde{K}_{\mu\nu}$, we conclude that if $\tilde{K}^+_{\mu\nu}$ is a solution for a given matter distribution, then there always exists another solution $\tilde{K}^-_{\mu\nu} = \eta_{\mu\nu} - \tilde{K}^+_{\mu\nu}$ for the same matter distribution, and the stability properties of the two solutions are opposite. Of course, since by definition $\tilde{K}_{\mu\nu} = -\frac{1}{\Lambda^3}\partial_\mu\partial_\nu\hat{\pi}$, we have to check that $\tilde{K}^-_{\mu\nu}$ is indeed the second derivative of a scalar field. However this is automatic, since $\eta_{\mu\nu}$ itself is a second derivative, $\eta_{\mu\nu} = \frac{1}{2}\partial_\mu\partial_\nu(x_\alpha x^\alpha)$.

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4. Classical stability

We now address in detail the issue of classical stability. We will show that under very general conditions a solution which becomes trivial at spatial infinity is stable everywhere. Before proving this fact we want to specify what we mean by 'stable'. In particular, we are going to characterize the stability of a generic classical solution by studying the tensor structure of the quadratic action of the fluctuations, eq. (3.14), around that solution. Notice that while for a stationary solution the concept of stability is perfectly defined, for a solution which evolves in time such a definition is a bit tricky. After all, if a solution depends on time the energy of the fluctuations around that solution is not conserved. For instance, one can expect to observe resonance phenomena associated to modes whose proper frequency is of the order of the inverse time scale on which the background solution evolves. A resonant growing fluctuation can be certainly seen as an instability, but its presence is not simply encoded in the tensor structure of the quadratic action. Our intention can therefore appear to be naive. However, the point is that our characterization is certainly sensible for ultraviolet modes, well above all possible resonances, on time and length scales much shorter than the typical time and length scales on which the background varies. In such a regime it is perfectly acceptable to study the stability of the system as if $Z_{\mu\nu}(x)$ were constant in space and time. We call this concept of stability 'local', since it refers to the local structure of the quadratic lagrangian of the fluctuations, and in principle nothing prevents the system from being stable in a given spacetime region (i.e. stable against short wavelength fluctuations localized in that region), and unstable in a different region. In this section we prove that under proper although general conditions this cannot happen.

In a system described by a quadratic lagrangian of the type of eq. (3.14) two physically different kinds of instabilities must be considered. The first is similar to a tachyonic instability, and, roughly speaking, is related to the relative signs of the terms involving time and space derivatives. If one of the relative signs is wrong, there exists an exponentially growing mode. More precisely, consider the equation of motion deriving from eq. (3.14),

$$Z_{\mu\nu}\,\partial^\mu\partial^\nu\varphi = 0\,, \tag{4.1}$$

where, with the above discussion in mind, we neglected the space-time dependence of $Z_{\mu\nu}$. For a mode with four-momentum $k^\mu = (\omega, \vec{k})$, this gives the dispersion relation $Z_{\mu\nu}k^\mu k^\nu = 0$, which can be cast in the form

$$(Z_{00}\,\omega + Z_{0i}\,k^i)^2 = (Z_{0i}Z_{0j} - Z_{00}Z_{ij})k^i k^j\,. \tag{4.2}$$

In order for the system to be stable, the solution for ω must be real for any wave vector $\vec{k}$. The requirement is therefore that the $3{\times}3$ matrix $Z_{0i}Z_{0j} - Z_{00}Z_{ij}$ is positive definite. Although not manifestly, this condition is clearly Lorentz invariant, since it corresponds to the requirement that the Lorentz invariant equation $Z_{\mu\nu}k^\mu k^\nu = 0$ has only real solutions (apart from an arbitrary overall phase multiplying k^μ).

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The second kind of instability one has to consider is the ghost-like instability, which is related to the overall sign of the kinetic lagrangian. Suppose that all relative signs are correct, so that all frequencies are real and there exists no exponentially growing mode. However, if the overall sign of the quadratic lagrangian is wrong and our scalar field is coupled to ordinary matter, then it is possible with zero total energy to amplify fluctuations in both sectors, and this process is going to happen spontaneously already at the classical level. At the quantum level the problem is probably even worse, although it is not yet clear how to do sensible and consistent computations in a quantum theory with ghosts in the physical spectrum (see for instance [22, 23]). This kind of instability, unlike the previous, is there only in presence of interactions with other sectors. If the system is free of tachyonic instabilities, i.e. if $Z_{0i}Z_{0j} - Z_{00}Z_{ij}$ is positive definite, then the ghost-like instability is avoided if $Z_{00} > 0$.

In the hamiltonian formalism both types of instability, although physically different, are formally on an equal footing. Their absence is guaranteed whenever the hamiltonian is positive definite. For our action, eq. (3.14), the hamiltonian density of the fluctuations reads

$$\mathcal{H}(\Pi, \varphi) = \frac{1}{4Z_{00}}(\Pi + 2Z_{0i}\,\nabla_i\varphi)^2 - Z_{ij}\,\nabla_i\varphi\nabla_j\varphi\,, \tag{4.3}$$

where Π is the conjugate momentum of φ, that is $\Pi = 2(Z_{00}\,\dot\varphi - Z_{0i}\,\nabla_i\varphi)$. $\mathcal{H}$ is positive definite for positive Z_{00} and negative definite Z_{ij}. The latter is a stronger requirement than the one we found above, since $Z_{0i}Z_{0j}$ is always a positive definite matrix. However here there is a subtlety. What must be positive definite is not the hamiltonian density, but rather its integral in d^3x, i.e. the total hamiltonian. The mixed term $\sim \Pi\nabla_i\varphi$ that comes from expanding the square in eq. (4.3) is conserved on the equation of motion, once integrated in d^3x, since its time derivative turns out to be a total gradient. Therefore the integral $H_0 = \int d^3x\mathcal{H}_0$, with

$$\mathcal{H}_0(\Pi, \varphi) = \frac{1}{Z_{00}}\left[\frac{1}{4}\Pi^2 + (Z_{0i}Z_{0j} - Z_{00}Z_{ij})\nabla_i\varphi\nabla_j\varphi\right], \tag{4.4}$$

is conserved on the equations of motion of $H = \int d^3x\mathcal{H}$. Although H_0 is not the hamiltonian, a sufficient condition for stability (compact orbits) is that H_0 be positive definite. Notice that this condition is weaker than that set by positivity of H, but it coincides with the condition found from the field equations. In summary, in order for a solution to be stable one needs that Z_{00} be positive and that the matrix $Z_{0i}Z_{0j} - Z_{00}Z_{ij}$ be positive definite.

Consider now a generic distribution of matter sources for the π field. We stick to the case in which matter can be described by a fluid. Its energy-momentum tensor is thus of the form $T_{\mu\nu} = (\rho + p)\,u_\mu u_\nu + p\,\eta_{\mu\nu}$, where ρ is the rest-frame energy density, p is the rest-frame pressure, and u^μ is the fluid four-velocity. The π field is sensitive only to the trace $T = T^\mu{}_\mu = -(\rho - 3p)$, which does not depend on u^μ. We consider the case in which the sources are localized, i.e. their energy-momentum tensor decays at spatial infinity. In such a case far away from the source the trivial configuration $\tilde K_{\mu\nu} = 0$, $Z_{\mu\nu} = -3\eta_{\mu\nu}$ is a solution of the field equations. Moreover, we start with the assumption that $p = 0$ and

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$\rho \geq 0$, although we will relax it below. We will show that under these hypotheses the system is stable.

Fix a generic x^μ. $Z_{\mu\nu}$ is a (x-dependent) symmetric tensor, therefore we have some hope of diagonalizing it with a (x-dependent) Lorentz transformation. This is not always possible, since the Minkowski metric $\eta_{\mu\nu}$ is not positive definite, and the spectral theorem does not apply. In the Appendix we show that $Z_{\mu\nu}$ can be certainly diagonalized with a Lorentz transformation if the typical velocities of the matter sources are non-relativistic. This is plausible from a physical point of view, since, roughly speaking, the failure of the spectral theorem can be traced to a possible strong mixing between space and time that can lead to a zero-norm eigenvector of $Z^\mu{}_\nu$. For relativistic motions of the sources we are not guaranteed that the diagonalization is possible. In order to proceed we have to assume it, so the result we are going to find can be applied only if such a diagonalization is possible. Let's therefore postulate that there exists a Lorentz frame in which $Z^\mu{}_\nu = \mathrm{diag}(z_0, z_1, z_2, z_3)$. In such a frame eq. (3.13) reads

$$F(z) \equiv -\frac{2}{3}\left[(z_0^2 + \cdots + z_3^2) - (z_0 z_1 + z_0 z_2 + \cdots + z_2 z_3)\right] = 6 + \frac{\rho}{2\Lambda^3 M_4} \geq 6\,. \tag{4.5}$$

In order for the system to be locally stable in x^μ we want a positive Z_{00} and a positive definite $Z_{0i}Z_{0j} - Z_{00}Z_{ij}$. In the Lorentz frame we are using this simply reduces to the requirement that all z_μ's are negative. In the space of the eigenvalues of $Z^\mu{}_\nu$ we can associate to this condition four critical hyperplanes, namely $z_\mu = 0$, for $\mu = 0, \ldots, 3$. On these hyperplanes the system is marginally stable. We want to show that these critical hyperplanes do not intersect the set of solutions of eq. (4.5). This result would permit us to conclude that the system is stable. In fact, at $\vec{x} \to \infty$ the system is locally stable, since the $\tilde{K}_{\mu\nu} = 0$ configuration is stable. It could become unstable in moving to finite $\vec{x}$, but by continuity its trajectory in the eigenvalues space should cross one of the four critical hyperplanes. If the intersection between the critical hyperplanes and the set of solutions is null, this cannot happen.

The proof of the above fact is straightforward. Take a critical hyperplane, for instance $z_0 = 0$, and impose it as a constraint on the l.h.s. of eq. (4.5). One gets

$$\begin{aligned} F(z)|_{z_0=0} &= -\frac{2}{3}\left[(z_1^2 + z_2^2 + z_3^2) - (z_1 z_2 + z_1 z_3 + z_2 z_3)\right] \\ &= -\frac{1}{3}\left[(z_1 - z_2)^2 + (z_1 - z_3)^2 + (z_2 - z_3)^2\right] \leq 0\,, \end{aligned} \tag{4.6}$$

while in order to solve eq. (4.5) $F(z)$ should be larger or equal than 6. In conclusion, the z_μ's that lie on a critical hyperplane cannot satisfy the field equation.

Of course, this argument does not prove that unstable solutions do not exist. Actually, as we showed in the previous section, for every locally stable solution there exists a locally unstable conjugate solution. What this line of reasoning proves is that the sets of locally stable and locally unstable solutions are topologically disconnected in the eigenvalues space, and therefore a solution which is locally stable at one point (infinity, in our case) is stable everywhere. And vice-versa: a solution which is locally unstable at one point is unstable everywhere. Notice that in the above proof the requirement that the sources are localized

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was needed only to ensure that at infinity the trivial configuration $\tilde{K}_{\mu\nu} = 0$ is a solution of the field equation, so that there exists at least one point at which the system is locally stable. In the case of non-localized sources, if a specific solution is known by other means to be locally stable at some point, then its stability throughout the Universe is guaranteed by the same arguments as above.

We have shown that, for sources with zero pressure and positive energy density decaying at spatial infinity, the solution which is trivial at infinity is stable everywhere. We can relax all these requirements, in order to find a condition under which the sets of locally stable and locally unstable solutions in the eigenvalues space come into contact, so that a solution could in principle be stable somewhere and unstable somewhere else. Allowing for generic ρ and p, eq. (3.13) now reads

$$F(z) = 6 + \frac{(\rho - 3p)}{2\Lambda^3 M_4} . \tag{4.7}$$

From this and eq. (4.6) we see that a critical surface intersects the set of solutions if and only if

$$(\rho - 3p) \leq -12\,\Lambda^3 M_4 = -12\,m^2 M_4^2 . \tag{4.8}$$

This is a necessary condition for the development of local instabilities, starting from a solution locally stable at some point, for instance at infinity.

So far we have dealt with very general features of the classical solutions of the DGP model. We now move to explicitly derive two interesting solutions and study their properties.

5. de Sitter solution

In refs. [7, 8] it was shown that the large-distance modification of gravity implied by the DGP model can make a Friedmann-Robertson-Walker Universe accelerate even in absence of a cosmological constant. In particular, one finds that at late time, when all the energy density and pressure have redshifted away, the Universe approaches an accelerating phase with constant Hubble parameter $H = m$. In other words, a solution of the DGP model is a four-dimensional de Sitter space with curvature radius $L = 1/m$, embedded in a five-dimensional Minkowski bulk. The stability properties of this de Sitter solution were analyzed in ref. [11], where it was shown that around this configuration the π field has a ghost-like kinetic term.

In this section we show that the four-dimensional effective action of the π field, in absence of matter and cosmological constant, can account for this de Sitter solution. Of course, since de Sitter space is curved, while we are working in flat space, we have to restrict to a small patch around $x = 0$ and look at de Sitter space as a small deformation of Minkowski space. This is sensible for $x \ll L$, where L is the de Sitter curvature radius.

Consider eq. (3.4) in vacuum, that is set $T = 0$. We are looking for a maximally symmetric solution, therefore we consider the ansatz $\tilde{K}_{\mu\nu} = \frac{1}{4}\tilde{K}\eta_{\mu\nu}$. We obtain

$$\tilde{K} = 4 , \qquad \tilde{K}_{\mu\nu} = \eta_{\mu\nu} , \qquad \hat{\pi}(x) = -\frac{1}{2}\Lambda^3\, x_\mu x^\mu . \tag{5.1}$$

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In the language of section 3, this is the solution which is conjugate to the trivial configuration $\tilde{K}_{\mu\nu} = 0$. Therefore, we know that it must be unstable. The quadratic action of the small fluctuations φ around this solution in fact is

$$S_\varphi = \int d^4x \left[-3(\partial\varphi)^2 - 2\left(\tilde{K}_{\mu\nu} - \eta_{\mu\nu}\tilde{K}\right)\partial^\mu\varphi\partial^\nu\varphi\right] = \int d^4x\, 3(\partial\varphi)^2\,, \tag{5.2}$$

exactly reversed with respect to the 'canonical' one. This is precisely the result of ref. [11] about the kinetic term of the π field around the de Sitter background.

To check that this solution corresponds locally to de Sitter geometry we must take into account the mixing between π and the metric perturbation $h_{\mu\nu}$. This is straightforward to do by noticing that in terms of the Weyl-transformed metric $h'_{\mu\nu} = h_{\mu\nu} - \frac{1}{M_4}\hat{\pi}\eta_{\mu\nu}$ the quadratic lagrangian, eq. (2.4), is diagonal, i.e. there is no mixing between π and $h'_{\mu\nu}$. The solution we are studying corresponds to $h'_{\mu\nu} = 0$, so that the metric perturbation is

$$h_{\mu\nu} = \frac{1}{M_4}\hat{\pi}\eta_{\mu\nu} = -\frac{1}{2}m^2\,\eta_{\mu\nu}\,x_\alpha x^\alpha\,. \tag{5.3}$$

This describes locally a de Sitter space. In fact, by performing a gauge transformation with parameter $\epsilon_\mu = \frac{1}{4}m^2\,x_\mu x_\alpha x^\alpha$, the metric becomes

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} + \partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = m^2\,x_\mu x_\nu\,, \tag{5.4}$$

and this is exactly the induced metric near $x = 0$ on the hyperboloid

$$\eta_{\mu\nu}\,x^\mu x^\nu + y^2 = \frac{1}{m^2}\,, \tag{5.5}$$

which describes the embedding in five dimensional flat space of a four dimensional de Sitter space with curvature radius $L = 1/m = L_{\rm DGP}$.

We can check explicitly that in the regime of validity of the solution we have found all higher order interactions, eq. (2.7), can be consistently neglected. Our solution has $\hat{\pi} \sim \Lambda^3 x^2$, $N'_\mu = 0$ and $h'_{\mu\nu} = 0$, so that eq. (2.7) reduces to

$$\Delta\mathcal{L}_{\rm bdy} \sim mM_4^2\,\partial(m\,x)^q \sim m^2M_4^2\,(m\,x)^{q-1}\,, \tag{5.6}$$

to be compared with the cubic self-coupling of π, which on the solution is of order $\Lambda^6\,x^2 = m^2M_4^2\,(m\,x)^2$. We can trust our solution only for $x \ll L_{\rm DGP} = 1/m$, since we are working in a nearly flat space. In this regime eq. (5.6) gives a negligible contribution to the action.

6. Spherically symmetric solution

As pointed out in refs. [9, 10], one could in principle test the DGP model by solar system experiments, at distances much smaller than the DGP scale, which instead is of the order of the Hubble horizon. This is because the correction to the Newton potential of an isolated body becomes important at distances of the order of the so-called Vainshtein scale,

$$R_V = \left(\frac{M}{m^2M_4^2}\right)^{1/3} \sim \left(R_S L_{\rm DGP}^2\right)^{1/3} \ll L_{\rm DGP}\,, \tag{6.1}$$

where M is the mass of the body and R_S its Schwarzschild radius. At smaller distances the relative importance of this correction is small, of order $(r/R_V)^{3/2}$, but nevertheless non negligible. An interesting fact is that, as shown in ref. [9], unlike its magnitude, the *sign* of this correction depends on the cosmological phase, i.e. on the behaviour of the Universe at cosmological distances.

In our formalism all the relevant features of the DGP model are encoded in the self-interacting dynamics of π. We will derive an explicit solution for the field generated by a point-like source, and show that its properties agree with the above results.

Consider a static point-like source of mass M, located at the origin: $T = -M\,\delta^3(\vec{x})$. We look for a static spherically symmetric solution $\hat{\pi}(r)$, where r is the radial coordinate. In such a case the field equations further simplifies; in particular eq. (3.2) becomes

$$\vec{\nabla}\cdot\left[6\,\vec{\nabla}\hat{\pi} - \frac{1}{\Lambda^3}\vec{\nabla}(\vec{\nabla}\hat{\pi})^2 + \frac{2}{\Lambda^3}\vec{\nabla}\hat{\pi}\,\nabla^2\hat{\pi}\right] = \vec{\nabla}\cdot\left[6\,\vec{E} - \frac{1}{\Lambda^3}\vec{\nabla}|\vec{E}|^2 + \frac{2}{\Lambda^3}\vec{E}\,\vec{\nabla}\cdot\vec{E}\right]$$
$$= \vec{\nabla}\cdot\left[6\,\vec{E} + \hat{r}\,\frac{4}{\Lambda^3}\frac{E^2}{r}\right] = \frac{M}{2M_4}\,\delta^3(\vec{x})\,, \tag{6.2}$$

where we defined $\vec{E} \equiv \vec{\nabla}\hat{\pi}(r) \equiv \hat{r}E(r)$, and we used $\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{d}{dr}(r^2\,E)$. After integration over a sphere centered at the origin, this reduces to an algebraic equation for $E(r)$,

$$4\pi r^2\left(6E + \frac{4}{\Lambda^3}\frac{E^2}{r}\right) = \frac{M}{2M_4}\,, \tag{6.3}$$

whose solutions are

$$E_\pm(r) = \frac{\Lambda^3}{4r}\left[\pm\sqrt{9r^4 + \frac{1}{2\pi}\,R_V^3\,r} - 3r^2\right]\,, \tag{6.4}$$

where R_V is the Vainshtein scale, eq. (6.1). The solution for $\hat{\pi}$ is obtained by integrating E along r, $\hat{\pi}_\pm(r) = \int E_\pm(r)dr$.

At small distances from the source the two solutions reduce to

$$E_\pm(r \ll R_V) = \pm\frac{\Lambda^3}{4\sqrt{2\pi}}\,\frac{R_V^{3/2}}{r^{1/2}}\,. \tag{6.5}$$

The acceleration induced on a test mass by π is roughly $\frac{1}{M_4}E$. The relative correction to the Newton force is thus of order

$$\frac{F_\pi}{F_{\rm Newton}} \sim \frac{E/M_4}{M/M_4^2 r^2} \sim \left(\frac{r}{R_V}\right)^{3/2}\,, \tag{6.6}$$

as expected. Notice that the magnitude of the correction does not depend on the solution chosen, but its sign does. This fact matches to what we mentioned at the beginning of this section, namely that the sign of the correction should depend on the behaviour of the Universe at cosmological distances. Indeed, although the two solutions $E_\pm$ are similar to each other near the source, far away they behave very differently: E_+ decays as $1/r^2$, while E_- blows up as r^2. Since E_+ at spatial infinity reduces to the trivial configuration $\tilde{K}_{\mu\nu} = 0$, one is tempted to conclude that E_- should approach the de Sitter solution found in the

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previous section, since the de Sitter and the trivial solutions are conjugate to each other. However, this is not the case, as one can easily check by noticing that E_- does not reduce to a maximally symmetric configuration at infinity. Indeed, E_+ and E_- are not conjugate in the sense of section 3, but rather they are 'spatially' conjugate, in the following sense. E_+ is constant in time, therefore it produces a purely spatial $\tilde{K}^+_{\mu\nu}$, that is $\tilde{K}^+_{\mu 0} = 0$. By the same arguments of section 3 one easily concludes that another static solution of the field equations is $\tilde{K}_{ij} = \frac{3}{2}\delta_{ij} - \tilde{K}^+_{ij}$. E_- corresponds precisely to this solution.

Since we are considering a localized source, with positive energy density, we know from the results of section 4 that the solution trivial at infinity, E_+, is stable everywhere against small fluctuations. We can check this fact explicitly. For $\hat{\pi} = \hat{\pi}(r)$, eq. (3.9) in spherical coordinates reads

$$S_\varphi = \int d^4x \sqrt{g} \left[\left(3 + \frac{2}{\Lambda^3} \nabla^2 \hat{\pi} \right) (\dot{\varphi}^2 - g^{ij}\, \partial_i \varphi \partial_j \varphi) + \frac{2}{\Lambda^3} \nabla_i \nabla_j \hat{\pi}\, g^{ik} g^{jl}\, \partial_k \varphi \partial_l \varphi \right], \tag{6.7}$$

where $g_{ij} = \mathrm{diag}(1, r^2, r^2 \sin^2\theta)$ is the metric tensor. The interesting non-zero components of the connection are $\Gamma^r_{\theta\theta} = -r$ and $\Gamma^r_{\phi\phi} = -r\sin^2\theta$, so that

$$\nabla_r \nabla_r \hat{\pi} = E' , \qquad \nabla_\theta \nabla_\theta \hat{\pi} = r\, E , \qquad \nabla_\phi \nabla_\phi \hat{\pi} = r \sin^2\theta\, E , \tag{6.8}$$

while $\nabla^2 \hat{\pi} = E' + \frac{2}{r} E$. In the end we have

$$\begin{aligned} S_\varphi = \int d^4x \sqrt{g} \Big\{ & \left[3 + \frac{2}{\Lambda^3} \left(E' + \frac{2\,E}{r} \right) \right] \dot{\varphi}^2 - \left[3 + \frac{4}{\Lambda^3} \frac{E}{r} \right] (\partial_r \varphi)^2 - \\ & - \left[3 + \frac{2}{\Lambda^3} \left(E' + \frac{E}{r} \right) \right] (\partial_\Omega \varphi)^2 \Big\} , \end{aligned} \tag{6.9}$$

where we defined $(\partial_\Omega \varphi)^2$ as the angular part of $(\vec{\nabla}\varphi)^2$,

$$(\partial_\Omega \varphi)^2 \equiv (\vec{\nabla}\varphi)^2 - (\partial_r \varphi)^2 = \frac{1}{r^2} (\partial_\theta \varphi)^2 + \frac{1}{r^2 \sin^2\theta} (\partial_\phi \varphi)^2 . \tag{6.10}$$

By direct inspection of the solution E_+ it is straightforward to see that the factors enclosed in square brackets are everywhere positive, thus proving the stability of the solution.

Similarly to the de Sitter case studied in the previous section, also on this solution higher order interactions of the type of eq. (2.7) give a negligible contribution to the action, even well inside the Vainshtein region, where the extrinsic curvature is getting larger and larger. To see this, it is sufficient to plug into eq. (2.7) the behaviour of the solution for $R_S \ll r \ll R_V$, namely $\hat{\pi} \sim \Lambda^3 R_V^{3/2} / r^{1/2}$, $h'_{\mu\nu} \sim R_S/r$ and $N'_\mu = 0$. The result is

$$\Delta \mathcal{L}_{\mathrm{bdy}} \sim \frac{m M_4^2}{R_S} \left(\frac{R_S}{r} \right)^{q/2+s+1} , \tag{6.11}$$

which clearly shows that as long as $r \gg R_S$ the dominant contribution comes from the cubic self-coupling of π, i.e. from the term with $q = 3$ and $s = 0$. Like in the de Sitter case, as long as the space is nearly flat the leading interaction term is the cubic self-coupling of π, in agreement with the general argument given in section 2.

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We want to conclude with a comment on the behaviour of the solution in the presence of several sources. Of particular interest for the discussion of the following section is the behaviour of the effective kinetic term coefficient $Z_{\mu\nu}$. In the Vainshtein region $Z_{\mu\nu}$, the extrinsic curvature $K_{\mu\nu}$ and the Riemann tensor $R^{\mu}{}_{\nu\rho\sigma}$ scale as

$$Z \sim KL_{\rm DGP} \sim \sqrt{R}\,L_{\rm DGP} \sim \sqrt{R_S}\,\frac{L_{\rm DGP}}{r^{3/2}}\,, \tag{6.12}$$

where here on we neglect the indices. Notice, in particular, that the extrinsic and Riemann curvatures are associated to the same length scale: $K \sim \sqrt{R}$. For the asymptotic linear field the relation was instead $K \sim RL_{\rm DGP} \ll \sqrt{R}$. Now, the relation $Z \sim \sqrt{R}\,L_{\rm DGP}$ is valid also in more general situations. For instance we have checked it explicitly in the case of a spherical matter distribution not localized at one point. It is also easy to derive it for the interesting case of two (several) pointlike sources, for instance the Sun and the Earth. As $M_\odot \gg M_\oplus$ we can neglect the Earth contribution in the asymptotic field. $M_\oplus$ becomes relevant only close enough to Earth. In this region we can approximately treat the Sun field as a background giving just rise to a wave function $Z_\odot \sim \sqrt{M_\odot}\,L_{\rm DGP}/(M_4 r_\oplus^{3/2})$ where $r_\oplus$ is the Earth-Sun distance. By defining a canonical field $\hat\pi_\odot = \sqrt{Z_\odot}\hat\pi$, the field equation close to Earth is just eq. (6.2) with renormalized values $\Lambda \to \Lambda_\odot \equiv \Lambda\sqrt{Z_\odot}$, $M_4 \to M_4\sqrt{Z_\odot}$. Then applying the previous results we have that the renormalized Vainshtein radius for the Earth field is $R_{\odot V}^{\,3} = M_\oplus/(M_4\Lambda^3 Z_\odot^2) = r_\oplus^3 M_\oplus/M_\odot$. This is precisely the distance from Earth at which the Riemann curvature becomes dominated by the Earth field. By applying the results of this section, at shorter distances the full $\hat\pi$ wave function is just $Z \sim Z_\odot(R_{\odot V}/r)^{3/2} \sim \sqrt{R}\,L_{\rm DGP}$, where R is now dominated by the Earth field. Notice in particular that $R_{\odot V}$ is larger than the Earth-Moon distance, so that the effects of the π field on the Moon rotation [9, 10] can be studied by neglecting the presence of the Sun in first approximation. This is precisely like in GR: the Sun's tidal force can be neglected when studying the Moon's orbit in lowest approximation.

7. Quantum effects

Our discussion was until now classical. In order to characterize quantum versus classical effects it is useful to keep $\hbar \neq 1$ and write the lagrangian eq. (3.1) in terms of the non-canonical field π and the 4 and 5D Newton's constants G_4 and G_5,

$$\mathcal{L} = -\frac{G_4}{G_5^2}(\partial\pi)^2 + \frac{1}{G_5}(\partial\pi)^2\partial^2\pi + \frac{G_4}{G_5}\pi T \tag{7.1}$$

(throughout this section we will neglect $\mathcal{O}(1)$ factors and be schematic with the contraction of derivative and tensor indices). At the classical level physical quantities do not depend on the reparametrizations $\pi \to \sigma\pi$ and also $\mathcal{L} \to \eta\mathcal{L}$, with σ, η constants. For instance, in the case of a localized source $T = M\delta^3(x)$, the only invariant quantity is precisely the Vainshtein scale $R_V^3 = MG_5^2/G_4$. Indeed we can also define a classical expansion parameter, measuring the size of classical non linearities, by taking the ratio of the second and first terms in the lagrangian $\alpha_{cl} = G_5\partial^2\pi/G_4 \sim \tilde{K}$. Notice that α_{cl} is invariant under the above

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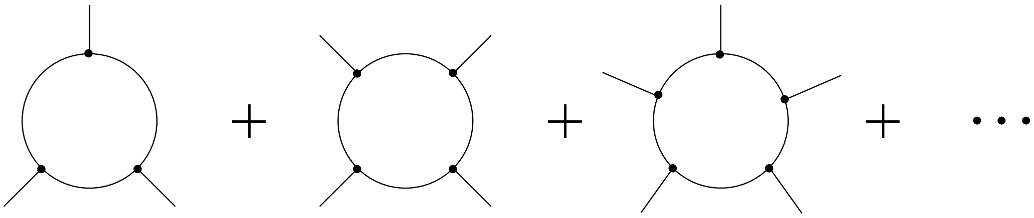

Figure 1: The graphs contributing to the 1-loop effective action, eq. (7.3).

reparametrizations. In the case of the spherical solution it is just a function of r/R_V. At the quantum level the lagrangian becomes a physical quantity, $\mathcal{L} \to \eta\mathcal{L}$ is no longer a 'symmetry', so that there is an additional physical length defined by $L_Q^6 = \hbar G_5^4/G_4^3$ or equivalently an energy $\Lambda = \hbar/L_Q$. For $L_{\rm DGP}$ of the order of the Hubble length we have $L_Q \sim 1000\,{\rm km}$. This analysis can be compared to the case of ordinary GR in 4D: the analogue of the Vainshtein scale is the Schwarzschild radius, while the classical expansion parameter is simply the gravitational perturbation $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$. Similarly, at the quantum level the Planck length is the analogue of L_Q.

Like we have defined a classical expansion parameter α_{cl} we can define a quantum expansion parameter $\alpha_q = (L_Q^2\partial\partial)$. In the case of a spherical potential we have $\alpha_q = (L_Q/r)^2$. Then we expect that any physical quantity A will be expandable with respect to the value A_0 it takes in the linearized classical limit as a double series

$$\frac{A - A_0}{A_0} = \sum_{n,m} a_{nm}\alpha_q^n \alpha_{cl}^m\,, \tag{7.2}$$

with n, m positive integers. The dependence on only integer powers of α_q follows from Lorentz symmetry. As an example of a quantity of interest we could consider the action.[3] As shown in ref. [11], and as evident from eq. (3.10), the loops of the quantum fluctuation field φ generate terms involving at least two derivatives on the external background, i.e. involving $\tilde{K}_{\mu\nu} = -\partial_\mu\partial_\nu\tilde{\pi}/\Lambda^3$ and its derivatives. In particular the tree level cubic interaction is not renormalized. For instance, cutting off the UV divergences at the scale Λ, the structure of the 1-loop correction is

$$\Gamma^{1-\rm loop} = \sum_m \left[a_m\Lambda^4 + b_m\partial^2\Lambda^2 + \partial^4(c_m \ln\Lambda + I_m)\right]\left(\frac{\partial\partial\tilde{\pi}}{\Lambda^3}\right)^m\,. \tag{7.3}$$

The graphs contributing to $\Gamma^{1-\rm loop}$ are depicted in figure 1; m counts the number of external legs, and by I_m we indicate the finite parts. Notice that $\Gamma^{1-\rm loop}/\mathcal{L}_{cl} = \Gamma^{1-\rm loop}/(\partial\pi)^2$ has the structure of eq. (7.2). We should stress however that the power divergent terms are strictly speaking not calculable, in the sense that they fully depend on the UV completion. On the other hand the log divergent terms are also associated with infrared effects and are thus calculable. They correspond in eq. (7.2) to powers of n that are multiple of 3: $\alpha_q^3 \sim \hbar\frac{G_5^4}{G_4^3}\partial^6 = E^6/\Lambda^6$ is the genuine loop counting parameter. For instance by working

[3]Notice that although the action is not a physical quantity at the classical level its relative variation $\delta S/S_0$ is physical, and indeed defines α_{cl}.

in dimensional regularization we would only get the minimal set of divergences that are required for the consistent definition of the theory, and these are precisely the logarithmic ones.

Now the important remark is that when the classical parameter α_{cl} is larger than $\mathcal{O}(1)$ the expansion in eq. (7.2) breaks down even if $\alpha_q \ll 1$. What basically happens is that large classical effects pump up the quantum corrections, and the result becomes fully dependent on the series of counterterms. Both power and log divergent terms at $\alpha_{cl} > 1$ formally swamp the classical action for a large enough number of external legs (m in eq. (7.2)). This implies that the details of the UV completion of DGP in general dominate the physics for $\alpha_{cl} > 1$ or at distances from a point source that are shorter than the Vainshtein radius R_V. Then we should not even be able to compute the gravitational potential at solar system distances, although they are still much bigger than the genuine quantum length L_Q. A similar problem has been pointed out in the case of massive gravity [15].

There are however indications that the situation may not be necessarily as bad as it seems. First, as we have already remarked, the problem arises when expanding in the number of external legs, that is when expanding in the classical background field. For any given amplitude with fixed number of legs the loop expansion, sum over n in eq. (7.2), is convergent as long as $r > L_Q$. Second the classical theory we are dealing with makes perfect sense even below the Vainshtein scale as we have proven in the previous sections. For a well defined class of energy-momentum sources and of boundary condition at spatial infinity, no pathology or ghostlike instability is encountered in the π system. Indeed even if the interaction term involves four derivatives, the quadratic action around the background remarkably contains quadratic terms that have only up to two derivatives: this prevents the appearance of ghostlike modes. This should be compared to what happens when higher derivative corrections to a kinetic term become important and ghost states appear: in those cases it is mandatory to assume that extra quantum effects eliminate the problem, or better to declare the regime of validity of the theory to be limited. Based on these two remarks we want to consider the structure of the quantum effective action more closely. As we did previously we decompose π into classical background and quantum fluctuation: $\pi = \pi_{cl} + \varphi$. The tree level lagrangian for the fluctuation is

$$\mathcal{L}_\varphi = Z_{\mu\nu}\partial^\mu\varphi\partial^\nu\varphi + \frac{1}{\Lambda^3}\partial^\mu\varphi\partial_\mu\varphi\Box\varphi \tag{7.4}$$

and $Z_{\mu\nu}$, see eq. (3.11), contains all the dependence on the classical background π_{cl}. In the regions where classical non-linearities are large, the eigenvalues of the kinetic matrix become large, $Z \sim \tilde{K} \gg 1$. Now, as long as Z does not lead to ghosts, a large Z will enhance the gradient energy and suppress quantum fluctuations in φ. This somewhat contradicts our previous conclusion that things are out of control when $\tilde{K} \gg 1$. In order to simplify the discussion and the notation consider a toy model where the background matrix reduces to a scalar $Z_{\mu\nu} \equiv -\eta_{\mu\nu}Z$. As long as Z varies slowly over space, it makes sense to define a local strong interaction scale $\tilde{\Lambda}(x) = \Lambda\sqrt{Z(x)}$, which describes the local scattering of quantum excitations φ. Of course this scale is obtained by writing the action in terms of a locally canonical field $\hat{\varphi} = \varphi \cdot \sqrt{Z}$, and the definition makes sense as long as the typical

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distance by which Z changes is itself much bigger than $1/\tilde{\Lambda}(x)$. It is easy to calculate the divergent part of the 1-loop effective action $\Gamma^{1-\text{loop}}$ by working with the canonical field $\hat{\varphi}$. The dependence from Z in $\Gamma^{1-\text{loop}}$ comes from two sources: 1) the jacobian of the rescaling in the path integral, 2) a spacetime dependent mass term $-m^2(x)\hat{\varphi}^2$ generated by the kinetic term after rescaling,

$$m^2(x) = \frac{1}{4}\left(\frac{\partial_\mu Z \partial^\mu Z}{Z^2}\right) - \frac{1}{2}\frac{\partial^2 Z}{Z}\,. \tag{7.5}$$

The contribution from the mass term corresponds to the usual Coleman-Weinberg effective potential

$$\Gamma_1^{\text{CW}} = \frac{1}{16\pi^2}\left(\Lambda_{\text{UV}}^2 m^2 - (m^2)^2 \ln \Lambda_{\text{UV}} + \hat{\Gamma}_1^{\text{CW}}\right), \tag{7.6}$$

where the finite part $\hat{\Gamma}_1^{\text{CW}}$ is expected to be a fairly complicated non-local expression that fully depends on the spacetime behaviour of m^2. For instance, in a central field where $Z \propto r^a$, like in the Vainshtein region, we have $m^2 \sim 1/r^2$ and we expect $\hat{\Gamma}_1^{\text{CW}} = -(m^2)^2 \ln r$ plus terms of order m^4, but without log enhancement. On the other hand the jacobian factor yields formally

$$\Gamma_1^{\text{J}} = -\frac{1}{2}\delta^4(0)\ln Z\,. \tag{7.7}$$

This term does not depend on the derivatives of Z, and should correspond to a quartic divergence. We can make this explicit by regulating the delta function as $\delta^4(0) = \Lambda_{\text{UV}}^4/(2\pi)^4$. Notice that in principle the ultraviolet cutoff Λ_{UV} can be as high as the local scale $\tilde{\Lambda}(x)$. To comment this results let us focus first on the logarithmic divergent term, as it is completely unambiguous. We see that when $Z \gg 1$ this term does not blow up, as one could have concluded by looking at the series expression eq. (7.3). The effects of a large Z are resummed in the effective mass m^2, in a form that depends only on the size of the relative derivatives $\partial^n Z/Z$. For instance in the Vainshtein region the logarithmic part leads to a correction to the classical action which is of order

$$\frac{1}{16\pi^2}\frac{1}{r^4}\ln(\Lambda_{\text{UV}} r)\,. \tag{7.8}$$

This should be compared with the classical action which is of order $\Lambda^4(\Lambda R_V)^2(R_V/r)^{5/2} = r^2\tilde{\Lambda}^6(r)$: we find that the quantum correction is subdominant for $r > r_c = 1/(\Lambda^4 R_V^3) = L_Q/(M_4 R_S)$. Indeed r_c is precisely defined by the equality $r_c = \tilde{\Lambda}^{-1}(r_c)$ as one should have expected. Notice that for the interesting cases of macroscopic sources r_c is much shorter that $L_Q = 1/\Lambda = 1000$ km. We will have more to say on this below. We have more comments to make on this simple 1-loop computation. Eq. (7.6), when expanded in powers of $\tilde{K} = Z - 1$, corresponds to the calculation of the c_m coefficients of eq. (7.3). This object describes the 1-loop RG evolution of the effective lagrangian. On the other hand the initial conditions for this evolution at some scale, for instance Λ, are determined by arbitrary local counterterms contained in the I_n. In general, as long as locality and the symmetries are preserved, there is little restriction on these terms from the effective lagrangian view point. In particular they could become large just when Z is large. For instance a function of the form $Z^N(\partial Z/Z)^4$ would be a perfectly acceptable counterterm, with a perfectly fine

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analytic and local expansion around the point $\tilde{K} = Z - 1 = 0$. For $N \gg 1$ a term like this would dominate the action below the Vainshtein scale. What distinguishes a damaging term like this from the mild effects that we have explicitly computed? The difference is very simple: this term depends on the absolute magnitude of Z, while the others don't. After the rescaling, the dimensionful parameters in the tree level $\hat{\varphi}$ lagrangian are given by $\tilde{\Lambda} \equiv \Lambda\sqrt{Z}$ and by the derivatives $\partial Z/Z \equiv 2\partial\tilde{\Lambda}/\tilde{\Lambda}$, $\partial^2 Z/Z \equiv \partial^2\tilde{\Lambda}^2/\tilde{\Lambda}^2$, etc. The operator in question cannot be written just in terms of these local scales since $Z^N = \tilde{\Lambda}^{2N}/\Lambda^{2N}$ and the original scale Λ does not influence local quantities in the tree approximation. We can then characterize in a fairly simple way the class of 1-loop counterterms with four derivatives (log divergences) for which perturbativity is preserved below the Vainshtein scale (in fact below L_Q): it is given by local polynomials of $\tilde{\Lambda}$, $\partial^2\tilde{\Lambda}$, etc., weighted by the appropriate powers of $\tilde{\Lambda}$ to match dimensionality. Nothing particularly interesting is learned by considering the structure of the power divergences. After all this is not surprising since they are totally scheme dependent. However the lesson we drew from the logarithmic ones is clear enough to allow us to formulate a simple consistent requirement, a conjecture, valid for all type of counterterms, and to all orders, such that the quantum corrections remain under control way below the Vainshtein scale. The requirement is that the only dimensionful quantity determining the local counterterms be the local scale $\tilde{\Lambda}(x)$,

$$\mathcal{L}_{\rm CT} = \tilde{\Lambda}(x)^4 F\left(\frac{\partial}{\tilde{\Lambda}}\right), \tag{7.9}$$

where F schematically represents any infinite polynomial in $\partial\tilde{\Lambda}$, $\partial^2\tilde{\Lambda}$, etc. Notice that the above counterterm lagrangian should be added with $\tilde{\Lambda}$ written in terms of the physical field $\pi = \pi_{cl} + \varphi$, as this decomposition is arbitrary. Then we need to check that eq. (7.9) does not lead to new interactions for the quantum field φ associated to scales lower than $\tilde{\Lambda}$. If that were the case our ansatz would not be not self consistent since beyond 1-loop it would introduce a new scale in the effective action for the background field. However it is easy to verify that this consistency check is satisfied. Notice that the minimal set of counterterms that are needed in DR with minimal subtraction belongs to this class. However our definition is more general. For instance a term with no derivative $\tilde{\Lambda}^4$ is consistent with eq. (7.9) but is not generated in DR, since it corresponds to a quartic divergence. The physical meaning of our ansatz is also clear. The scale $\tilde{\Lambda}$ truly represents the scale at which new degrees of freedom come in. The UV divergences in the low energy effective theory are cut off at this scale. For instance the vacuum energy is essentially $\tilde{\Lambda}^4$ up to small derivative corrections associated to the space-time variation of $\tilde{\Lambda}$. From the point of view of the fundamental UV theory the field $Z = 1 + \tilde{K}$ is a sort of dilaton that controls the mass gap $\tilde{\Lambda}$. At the points where Z goes through zero the effective description breaks down since the gap vanishes and new massless states become relevant. This is another way to argue that the points where Z is negative, the ghost background, are outside the reach of our effective field theory and should be discarded.

We based all our discussion on the simplified case $Z_{\mu\nu} = -Z\eta_{\mu\nu}$, in which a rescaling of φ allowed us to define the running scale $\tilde{\Lambda}(x)$. In the general case this cannot of course be done. The procedure then is to work at each point with a short distance expansion of

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the φ propagator,

$$\langle\varphi(x+y)\varphi(x-y)\rangle = \frac{1}{Z_{\mu\nu}(x)y^\mu y^\nu} + d_1\frac{y^\rho y^\mu y^\nu \partial_\rho Z_{\mu\nu}(x)}{(yZy)^2} + \cdots\,, \tag{7.10}$$

where the dimensionless coefficients (functions of x) $d_1,\ldots$ are calculated in perturbation theory around a slowly varying Z background. In spite of this more complicated tensor structure the discussion is qualitatively the same: the $1/Z$ in the propagator arrange with the $1/\Lambda^3$ in the vertices to yield a sort of running scale $\tilde\Lambda(x)$. However the dynamics at the scale $\tilde\Lambda$ will not be Lorentz invariant. Anyway we stress that in the simple interesting case of a spherical source inside the Vainshtein region we have

$$Z_{\mu\nu} \simeq \left(\frac{R_V}{r}\right)^{3/2} A_{\mu\nu}\,, \tag{7.11}$$

where $A_{\mu\nu}$ is a constant matrix. Then a simple rescaling of the quantum field is sufficient to study the effective action.

At this point one will wonder about naturalness. Are we advocating an incredibly tuned structure of counterterms not dictated by any symmetry but just by our desire to save the model? From the point of view of the effective theory expanded around $\pi = 0$ it looks like we are making a big tuning. This was the viewpoint of ref. [11] for DGP, and, earlier, of ref. [15] for massive gravity. However, as we have proven in the first part of this paper, the validity of the classical theory extends well beyond the point $\pi = 0$. By validity of the classical theory here we include the fact that $\tilde\Lambda$ never crosses 0. All we have done in this section is to demand uncalculable UV effects not to drastically perturb this picture throughout the patch in field space where the classical theory works. The fact that this patch is sensibly bigger than the point $\pi = 0$ leads to stronger constraints on the counterterms. But when looking at the region $\pi \neq 0$ there is more to say. Around the points with $\partial\partial\hat\pi \gg \Lambda^3$ our ansatz can indeed be motivated by a symmetry: scale invariance.[4] In the limit $\partial\partial\hat\pi \gg \Lambda^3$ we can neglect the quadratic kinetic term and work just with the cubic lagrangian at tree level. The cubic lagrangian is obviously scale invariant by assigning $\hat\pi$ dimension zero. We can also trivially absorb the powers of Λ in the $\hat\pi$ field: $\hat\pi = \Lambda\tilde\pi$. The background $\tilde\pi_{cl}$ spontaneously breaks scale invariance. In particular it generates the mass scale $\tilde\Lambda^2 \sim \partial\partial\tilde\pi_{cl}$. Then it is obvious that our ansatz preserves scale invariance. The original quadratic kinetic term now reads $\Lambda^2(\partial\tilde\pi)^2$ representing a soft explicit breaking of scale invariance. When treating this term as a small perturbation it is then natural to generalize our eq. (7.9) to include all the terms proportional to positive powers of $\Lambda^2/\tilde\Lambda^2$. Now, one should be aware that scale invariance, unlike other global symmetries, cannot be readily used to naturally enforce parameter choices. Basically this is because conformal invariance in the interesting cases is never exact in the far UV. For instance quantum gravity breaks it. This limits the use of conformal symmetry as a substitute of supersymmetry in attacking the gauge hierarchy problem (see for instance a discussion in ref. [24]). However one can imagine a situation where the scale M at which a models flows to a conformal

[4]We thank Sergei Dubovsky for waking us up on this.

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point is much smaller than the physical and conformal breaking cut-off $\Lambda_{\rm UV}$. Then even though at the fixed point there may exist relevant deformations their coefficient will be naturally small (in units of M) if the corresponding operator is irrelevant at the UV scale. An example could be a 4-fermion term suppressed by $1/\Lambda_{\rm UV}^2$ in the microscopic theory, which becomes relevant at the fixed point thanks to a big anomalous dimensions. Such an example suggests that a theory may remain naturally scale invariant in a wide range of scales below the (conformal breaking) cut-off. In our case conformal symmetry is surely broken at the scale M_5, but we do not need to go that high in energy. This keeps open the possibility that our ansatz may indeed be technically natural. It would be interesting to investigate these issues in more detail, perhaps trying to come up with an explicit realization of a conformally invariant completion.

Finally we want to discuss the physical implications of our results. The running scale close to a spherical source can also be written as

$$\tilde{\Lambda}(r) = M_5^{1/2} R_S^{1/4} r^{-3/4}\,, \tag{7.12}$$

from which the critical length r_c at which the classical background π_{cl} stops making sense is just $r_c = 1/R_S M_5^2$. We have $r_c < R_S$ for $R_S > 1/M_5 \sim 1$ fm, so that the quantum effects in the field π have no implication on any relevant astrophysical black hole. Of course we still have to study the classical effects of the field π on 4D black holes.

In the presence of several macroscopic sources, like in the solar system, eq. (7.12) is dominated by the one generating the biggest gravitational tidal effect, i.e. the one dominating the Riemann tensor $\sim R_S/r^3$. At the surface of the Earth it is the Earth field itself that dominates $\tilde{\Lambda}$ and we find

$$\tilde{\Lambda}_\oplus^{-1} \sim 1\,\text{cm}\,. \tag{7.13}$$

In principle this is the length scale below which we cannot compute the gravitational potential at the surface of the Earth! Notice that this scale exceeds by almost 2 orders of magnitude the present experimental bound on modifications of gravity [25, 26, 27]. Notice also that we cannot play with M_5 in order to increase $\tilde{\Lambda}_\oplus$ by 10^2. This is because $\tilde{\Lambda}_\oplus \propto \sqrt{M_5}$ while $L_{\rm DGP} \propto 1/M_5^3$ so that we would lower at the same time $L_{\rm DGP}$ down to the unacceptably low value of ~ 1 kpc. However the 1 cm scale we are discussing is the one at which π, not the usual graviton, goes into a quantum fog. The graviton at this scale is still very weakly coupled. Moreover the field π at distances just a bit bigger that 1 cm is a very small source of macroscopic gravity, its contribution relative to the ordinary one being of order

$$\left(\frac{R_\oplus}{R_V}\right)^{3/2} \equiv \frac{\Lambda^2}{\tilde{\Lambda}_\oplus^2} \sim 10^{-16}\,. \tag{7.14}$$

Then it is reasonable to expect that whatever UV physics will take over below 1 cm it will not raise the effects of this sector on the gravitational potential by 16 orders of magnitude right away. Nonetheless this result may be viewed as yet another motivation to study the gravitational forces below a mm.

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8. Outlook

We conclude by briefly mentioning some directions in which our analysis could be extended. First of all, there is the issue of stability of time-dependent classical solutions. As we have already stressed, our proof of classical stability is rigorously valid only for matter sources with non relativistic velocities. It would be interesting to understand to what extent the stability is guaranteed in the presence of relativistic sources. Perhaps one can encounter astrophysical or cosmological situations in which the kinetic lagrangian of the fluctuation around the background passes through zero in some spacetime region. Such a region would be characterized by a local scale $\tilde{\Lambda} = 0$. This clearly indicates a breakdown of the effective theory, and in principle one expects unsuppressed quantum effects to take over, with interesting phenomenological signatures. Of course these are, by definition, out of the reach of the effective theory, and we can say nothing about them without a UV completion of the model.

On the other hand it would also be interesting to study whether the approximate conformal symmetry of our ansatz can tell us more about the UV completion. Notice in passing that by neglecting the original quadratic kinetic term, the π-sector becomes a critical[5] ghost condensate model. It is the presence of a classical energy-momentum source, and not the π self-interaction, that causes the π to condense, giving thus rise to a well defined effective field theory.

Another point which may require extra study concerns 4D black holes. Are they different from those of ordinary GR? The fact that the field π, when approaching the Schwarzschild radius, gives a more and more negligible correction to the Newton force suggests that black holes should be unaffected by its presence. However the very expansion we use breaks down close to the Schwarzschild radius, as shown by eqs. (2.7) and (6.11). Therefore a (numerical) study of the exact solution may be required. Of course the problem we are encountering here is just that close to the horizon we cannot consider the brane as approximately flat. A better viewpoint is gained by considering the full Einstein equations at the boundary in eq. (3.6). In the region $R_S \ll r \ll R_V$ the extrinsic curvature term, which controls deviations from GR, represents a small perturbation. This can be phrased as

$$R(g) \sim \sqrt{R_{\mu\nu}R^{\mu\nu}} \sim \frac{K}{L_{\rm DGP}} = \frac{1}{L_{\rm DGP}}\sqrt{\frac{R_S}{r^3}} \ll \sqrt{R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}} \sim \frac{R_S}{r^3}\,, \tag{8.1}$$

corresponding to the fact that the geometry is approximately Schwarzschild. The only way this state of things can drastically change is if the extrinsic curvature develops a singularity while approaching the Schwarzschild region. Without an explicit computation we cannot exclude this possibility, but we find it very unlikely. In analogy with GR, the horizon should not be a special locus from the point of view of the curvature.

Finally, it would be worth analyzing in more detail the connection between DGP and massive gravity, and investigating the possibility that our results, or some of them, can be exported to the latter. This is far from trivial, since massive gravity apparently doesn't share with DGP two remarkable features that turned out to be crucial in our analysis: the

[5]The kinetic term is not negative but zero.

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fact that the equation of motion for the strongly interacting field depends *algebraically* on the coefficients of the quadratic action around the background, and the absence in that action of terms with more than two derivatives.

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Note added. In a recent paper ref. [28] it was also argued that the UV completion of DGP should not destroy predictivity. Such a claim is in the same spirit of our paper, although ref. [28] is mostly based on an interesting σ-model version of DGP, where there are just scalar fields. In that toy model there is no analogue of our π field, with all its associated peculiarities, so that it is not immediately clear to us that there is a direct connection to our work. It would be interesting to further investigate this possibility.

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A. Diagonalization of $Z_{\mu\nu}$ in the non-relativistic limit

In this Appendix we want to prove that whenever the matter sources have non-relativistic velocities the tensor $Z_{\mu\nu}$ is diagonalizable with a Lorentz transformation.

First, from the very definition of $Z_{\mu\nu}$, eq. (3.11), we see that if $\tilde{K}_{\mu\nu}$ is diagonalizable, so is $Z_{\mu\nu}$. Given $\tilde{K}_{\mu\nu}$ in a generic frame, by means of a spatial rotation we can always align the spatial vector $\tilde{K}_{0i}$ to the x axis and cast $\tilde{K}_{\mu\nu}$ in the form

$$\tilde{K}_{\mu\nu} = \begin{pmatrix} k_0 & k_{01} & 0 & 0 \\ k_{01} & k_1 & \star & \star \\ 0 & \star & \star & \star \\ 0 & \star & \star & \star \end{pmatrix} . \tag{A.1}$$

Now, the point is that if by means of a boost along x we succeed in diagonalizing the upper-left 2×2 block we are done, since then by a spatial rotation we can diagonalize the full matrix. The problem is thus reduced to the diagonalization of a 2×2 symmetric tensor with a boost, i.e. to find two real eigenvectors of the matrix

$$\tilde{K}^{\alpha}{}_{\beta} = \begin{pmatrix} -k_0 & -k_{01} \\ k_{01} & k_1 \end{pmatrix} , \tag{A.2}$$

orthonormal with respect to the 2D Minkowski metric $\eta_{\alpha\beta}$ (now the indices α and β run over 0 and 1). The eigenvalues of $\tilde{K}^{\alpha}{}_{\beta}$ are

$$\lambda_{\pm} = \frac{1}{2}(k_1 - k_0) \pm \frac{1}{2}\sqrt{(k_1 + k_0)^2 - 4k_{01}^2} \, , \tag{A.3}$$

which are real and distinct as long as the off-diagonal element is small, $|k_{01}| < \frac{1}{2}|k_1 + k_0|$. In such a case there exist two real eigenvectors, orthonormal with respect to $\eta_{\alpha\beta}$. When instead $|k_{01}| \to \frac{1}{2}|k_1 + k_0|$ the two eigenvalues become equal, there is only one eigenvector and it has zero norm. Beyond this point, when $|k_{01}| > \frac{1}{2}|k_1 + k_0|$, both the eigenvalues and the eigenvectors are complex.

Now we want to make contact with the physics. The tensor $\tilde{K}_{\mu\nu}$ is related to the second derivatives of π, $\tilde{K}_{\mu\nu} = -\frac{1}{\Lambda^3}\partial_\mu\partial_\nu\pi$, so that

$$k_0 = -\frac{1}{\Lambda^3}\ddot{\pi}\,, \qquad k_{01} = -\frac{1}{\Lambda^3}\partial_x\dot{\pi}\,, \qquad k_1 = -\frac{1}{\Lambda^3}\partial_x^2\pi\,. \tag{A.4}$$

If the matter sources are stationary we can certainly choose π to be independent on time. In such a case $k_0 = k_{01} = 0$. Now we can look at the case in which the sources are evolving with non-relativistic velocities as a perturbation of the stationary case. In particular, if the typical velocity of the sources is v, then we expect that the time derivatives of π will be suppressed by powers of v with respect to the spatial gradients, namely

$$k_0 \sim v^2\, k_1\,, \qquad k_{01} \sim v\, k_1\,. \tag{A.5}$$

In the non-relativistic case, $v \ll 1$, the two eigenvalues in eq. (A.3) are certainly real and distinct,

$$\lambda_+ = k_1 - \frac{k_{01}^2}{k_1} + \mathcal{O}(v^3\, k_1)\,, \qquad \lambda_- = -k_0 + \frac{k_{01}^2}{k_1} + \mathcal{O}(v^3\, k_1)\,. \tag{A.6}$$

This shows that for non-relativistic sources the 2×2 matrix $\tilde{K}^\alpha{}_\beta$ is diagonalizable with a Lorentz boost, so that the full 4×4 $\tilde{K}_{\mu\nu}$ is diagonalizable, and so is $Z_{\mu\nu}$.

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